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Enumerative aspects of certain subclasses of perfect graphs

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Abstract

We investigate the enumerative aspects of various classes of perfect graphs like cographs, split graphs, trivially perfect graphs and threshold graphs. For subclasses of permutation graphs like cographs and threshold graphs we also determine the number of permutations π of $\{1,2,\dots,n\}$ such that the permutation graph $G[\pi]$ belongs to that class. We establish an interesting bijection between permutations whose permutation graphs are cographs (P_4-free graphs) and permutations that are obtainable using an *output-restricted deque* (Knuth, Art of Computer Programming, Vol I, Fundamental Algorithms) and thereby enumerate such permutations. We also prove that the asymptotic number of permutations of $\{1,2,\dots,n\}$ whose permutation graphs are split graphs is $\Theta(4^n/\sqrt{n})$. We also introduce a new class of graphs called C_5-split graphs, characterize and enumerate them. C_5-split graphs form a superclass of split graphs and are not necessarily perfect. All the classes of graphs that we enumerate have a finite family of small forbidden induced subgraphs.

Keywords: Generating functions; Isomorphism; Perfect graph; Output-restricted deque; Permutation graph; Cograph; Young tableaux; Polya's enumeration theory

1. Introduction

All graphs we deal with are finite, simple and undirected. As usual for a graph G, $V(G)$, $E(G)$, $\omega(G)$, $\alpha(G)$ and $\chi(G)$ respectively denote the vertex set, edge set, size of a largest clique, size of a maximum independent set and the vertex chromatic number. For $A\subseteq V(G)$, $G[A]$ stands for the subgraph of G induced by A. C_n stands for the cycle on n vertices, K_n for the complete graph on n vertices and P_n for the path on n vertices.

A graph G is said to be perfect if for every $A\subseteq V$, $\omega(G[A])=\chi(G[A])$; or equivalently G is perfect if $\alpha(G[A])=\theta_0(G[A])$ where $\theta_0(H)$ stands for the minimum number

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of cliques in H needed to cover the vertices of H, as shown by Lovasz [13]. For a comprehensive treatment of various classes of perfect graphs, see [6].

Two graphs G and H are *isomorphic* if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. Isomorphism is an equivalence relation and therefore partitions any set of graphs into isomorphism classes. A graph G with $|V(G)|=n$ is *labelled* if the n vertices are distinguished from one another by *names* such as $1,2,\dots,n$. For a class $\mathscr{C}$ of graphs, by enumeration of class $\mathscr{C}$ we generally mean determining the number of non-isomorphic and labelled graphs on n vertices belonging to the class $\mathscr{C}$. The number of labelled graphs in $\mathscr{C}$ with n vertices is the number of graphs in $\mathscr{C}$ with vertex set $\{1,2,\dots,n\}$, and by the number of unlabelled or non-isomorphic graphs in $\mathscr{C}$ we mean the number of isomorphism classes of graphs in $\mathscr{C}$ with n vertices.

Here we enumerate several subclasses of perfect graphs which have forbidden subgraph (induced) characterizations: split graphs (no $C_4, 2K_2$ or C_5), cographs (no P_4), threshold graphs (no P_4, C_4 or $2K_2$), and trivially perfect graphs (no P_4 or C_4). We also introduce a class of graphs called C_5-split graphs (no C_4 or $2K_2$), characterize and enumerate them. Wherever pertinent, we also enumerate permutation representations of a class of graphs (defined later). As a by-product, we also enumerate permutations that can be obtained using an *output-restricted deque* [11] by establishing a bijection between such permutations and P_4-free permutations (i.e permutations whose permutation graphs are cographs). We also determine the asymptotic value of the number of permutations of $\{1,2,\dots,n\}$ whose permutation graphs are split.

The (ordinary) generating function of a sequence $\{a_n\}_{n\geqslant 0}$ is the power series $\sum_{n=0}^{\infty} a_n x^n$. The exponential generating function of a sequence $\{a_n\}_{n\geqslant 0}$ is the series $\sum_{n=0}^{\infty} a_n x^n/n!$.

Lemma 1.1 (Harary and Palmer [7]). *If* $\sum_{m=0}^{\infty} A_m x^m = \exp\{\sum_{m=1}^{\infty} a_m x^m\}$, *then* $A_0 = 1$ *and for* $m\geqslant 1$, $A_m = m^{-1}(\sum_{k=1}^{m} k a_k A_{m-k})$.

2. Cographs

An undirected graph is a cograph if it has no induced P_4. If π is a permutation of the numbers $1,2,\dots,n$, then the graph $G[\pi]$ is defined as follows:

$$V = \{1,2,\dots,n\}$$

and

$$(i,j)\in E \quad \text{iff } (i-j)(\pi(i)-\pi(j)) < 0.$$

A graph G is called a permutation graph if there exists a permutation π such that $G \cong G[\pi]$. Note if π^{-1} is the inverse of the permutation π, $G[\pi] \cong G[\pi^{-1}]$. It is well known that all cographs are permutation graphs and that permutation graphs are perfect (see [6]).

Lemma 2.1. *If $G=(V,E)$ is a connected cograph, then there exist $u,v\in V$ such that $(u,v)\in E$ and $N_G(u)\cup N_G(v)=V$.*

Proof. Follows easily by induction. The base cases are trivial; let $G=(V,E)$ be a connected cograph on n vertices. Since G is connected it has a vertex v such that v is not a cut-vertex of G. By the induction hypothesis, since $H=G-v$ is a connected cograph on $n-1$ vertices, it has two vertices x,y such that $N_H(x)\cup N_H(y)=V-\{v\}$. Now, if either $(v,x)\in E$ or $(v,y)\in E$, x and y are the required vertices in G, so assume $(v,x),(v,y)\notin E$.

Since G is connected, $\exists w\in V$, $w\neq x,y$ such that $(v,w)\in E$. Considering the four vertices $\{v,w,x,y\}$ and noting that G is P_4-free we find that $(w,x),(w,y)\in E$. If one of the pairs $\{w,x\}$ or $\{w,y\}$ satisfy the requirements of the lemma, we are through; so assume they do not. In this case $\exists p,q\in V-\{v,w,x,y\}$ such that $(p,w),(p,y)\notin E$, but $(p,x)\in E$ and $(q,w),(q,x)\notin E$, but $(q,y)\in E$. Since G if P_4-free, we get considering the vertices $\{p,x,y,q\}$ that $(p,q)\in E$. But now, $\{w,x,p,q\}$ induces a P_4, a contradiction. □

Lemma 2.2. *If G is a connected cograph, then G^c is a disconnected cograph and vice versa.*

Proof. Let u,v be as specified by Lemma 2.1. Then using Lemma 2.1 and the fact that G^c has no induced P_4 (since P_4 is self-complementary), it is easy to see that u,v are in different connected components of G^c. □

2.1. Enumeration of non-isomorphic cographs

Let g_n (resp. c_n) denote the number of non-isomorphic cographs (resp. non-isomorphic connected cographs) on n vertices. Define $g_0=1$, $c_0=0$ and $g_1=c_1=1$. Let $G(x)=\sum_{m=0}^{\infty} g_m x^m$ and $C(x)=\sum_{m=0}^{\infty} c_m x^m$ be the corresponding generating functions.

Clearly by Lemma 2.2, $g_n=2c_n$ for $n\geqslant 2$ and hence we have

$$G(x)-1+x=2C(x). \tag{1}$$

Since G is a cograph iff all connected components of G are cographs, it follows that (see for instance [7]) $G(x)=\prod_{k=1}^{\infty}(1-x^k)^{-c_k}$ and so

$$\log G(x)=\sum_{k=1}^{\infty}\frac{C(x^k)}{k}. \tag{2}$$

By (1) and (2), we have

$$2C(x)-x+1=\exp\sum_{k=1}^{\infty}\frac{C(x^k)}{k}. \tag{3}$$

Using Lemma 1.1, we get, for $n \geqslant 1$,

$$g_n = n^{-1} \sum_{m=1}^{n} g_{n-m} \left(\sum_{d|m} d c_d \right) \quad \text{and} \quad g_m = 2c_m \quad \text{for } m \geqslant 2.$$

The above relations completely specify $\{g_n\}_{n \geqslant 0}$ and the enumeration is complete; we get $G(x) = 1 + x + 2x^2 + 4x^3 + 10x^4 + \cdots$.

The enumeration of labelled cographs is similar in spirit and it can be shown that the exponential generating function $f(x)$ for connected labelled cographs satisfies

$$f(x) = x + \mathrm{e}^{f(x)} - f(x) - 1.$$

However, since all cographs are permutation graphs, we now proceed to find the number of permutations π of $\{1, 2, \ldots, n\}$ such that $G[\pi]$ is a cograph.

2.2. Enumeration of permutation representations of cographs

Denote by p_n (resp. q_n) the number of permutations π of $\{1, 2, \ldots, n\}$ such that $G[\pi]$ is a cograph (resp. connected cograph). Also define, $p_0 = 1$, $q_0 = 0$ and $p_1 = q_1 = 1$. Using Lemma 2.2 and the fact that $G[\pi^{\mathrm{R}}] \cong G[\pi]^{\mathrm{c}}$ where π^{R} is the reversal of π, it follows that $p_n = 2q_n$ for $n \geqslant 2$. We require the following lemma for our enumeration:

Lemma 2.3. *Let $\pi = (a_1, a_2, \ldots, a_n)$ be a permutation of $\{1, 2, \ldots, n\}$ and $G = G[\pi]$. If C is a component with t vertices in G, then the vertices of C are $p, p+1, \ldots, p+t-1$ for some p, $1 \leqslant p \leqslant n - t + 1$.*

Proof. Let C be a connected component of G (consider C as a vertex set). If $|C| = 1$, the result is clear. Otherwise, let p, q be the smallest and largest vertices belonging to C.

Case 1: $(p, q) \in E$, then clearly $a_p > a_q$ since $p < q$. Now, for $p < i < q$ either $a_i > a_q$ or $a_i < a_p$, so either $(i, q) \in E$ or $(p, i) \in E$ which implies $i \in C$.

Case 2: $(p, q) \notin E$, so $a_p < a_q$. Suppose if possible let for some $p < i < q$, $i \notin C$. Then $a_p < a_i < a_q$. Now if $V_1 = \{j \in C \mid j < i\}$ and $V_2 = \{j \in C \mid j > i\}$, $V_1, V_2 \neq \emptyset$ as $p \in V_1$ and $q \in V_2$. Since C induces a connected subgraph, there exist $x \in V_1$ and $y \in V_2$ such that $(x, y) \in E$. Since $x < i < y$ it follows that either (x, i) or $(i, y) \in E$, which means that $i \in C$, a contradiction.

Thus in either case, $i \in C$ for $p \leqslant i \leqslant q$. The result now follows by the choice of p and q. $\square$

Theorem 2.1. *If $P(x) = \sum_{n=0}^{\infty} p_n x^n$, then $P(x) = \frac{1}{2}(3 - x - \sqrt{1 - 6x + x^2})$.*

Proof. In view of Lemma 2.3, for $n \geqslant 2$, a permutation π of $\{1, 2, \ldots, n\}$ such that $G[\pi]$ is a disconnected cograph is of the form $\pi = \pi_1 \pi_2$, where $\pi_1 = (a_1, a_2, \ldots, a_k)$ is permutation of $\{1, 2, \ldots, k\}$ for some $1 \leqslant k < n$ such that $G[\pi_1]$ is a connected cograph and π_2 is a permutation of $\{k+1,\dots,n\}$ such that $G[\pi_2]$ is a (possibly disconnected) cograph. Thus for $n \geqslant 2$,

$$p_n - q_n = q_n = \sum_{k=1}^{n-1} q_k\, p_{n-k}$$

or

$$p_n = \sum_{k=1}^{n} q_k\, p_{n-k} \quad (\text{since } p_0 = 1).$$

Note that since $p_1 = p_0 = q_1 = 1$, the above equation is valid even for $n = 1$ and thus we have

$$\sum_{n=1}^{\infty} p_n x^n = \left(\sum_{n=1}^{\infty} q_n x^n\right)\left(\sum_{n=0}^{\infty} p_n x^n\right)$$

or

$$P(x) - 1 = \tfrac{1}{2}(P(x) + x - 1)P(x) \quad (\text{since } p_n = 2q_n \text{ for } n \geqslant 2).$$

Solving the above equation and using the fact that $P(0) = 1$, we get

$$P(x) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}. \qquad \square$$

The coefficients $\{r_n\}_{n \geqslant 0}$ of the power series $R(x) = \sum_{n=0}^{\infty} r_n x^n$ where

$$R(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}$$

are called the *Schröder numbers* which made their original appearance in [19] and come up in several enumeration problems [16,17,20]. It is known that these numbers satisfy the recurrence relation

$$(n+1)r_n = 3(2n-1)r_{n-1} - (n-2)r_{n-2} \quad \text{for } n \geqslant 2$$

with $r_0 = 1$ and $r_1 = 2$. They also have an expression in terms of *Catalan* numbers $C_m = (1/(m+1))\,(2mm)\,0 = 1)$, as

$$r_n = \sum_{i=0}^{n} \binom{2n-i}{i} C_{n-i}.$$

In the case considered here $p_n = r_{n-1}$ for $n \geqslant 1$, thus providing yet another example of the occurrence of Schröder numbers in combinatorial problems.

2.3. Permutations obtainable using an output-restricted deque

2.3.1. Preliminaries

A permutation π such that $G[\pi]$ is a cograph will be called a P_4-free permutation. An output-restricted deque (OPD) is a linear list for which all insertions are made at either end of the list and deletions are made at only one end of the list. One may denote by s,q and x the operations of inserting an element at the left, inserting at the right and emitting an element from the left end of an OPD. A permutation π is obtainable using an OPD (called OPD-perm) if π can be obtained by a sequence of operations comprising of s,q and x performed using the numbers $1,2,\ldots,n$ in that order starting with the empty deque (see [11] for details). Let d_n denote the number of OPD-permutations on $\{1,2,\ldots,n\}$. In this section, we obtain a 'closed form' for the generating function $D(z)=\sum_{n\geqslant 0} d_n z^n$ by exhibiting a nice bijection from the set of P_4-free permutations on $1,2,\ldots,n$ to the set of OPD-permutations on $1,2,\ldots,n$. This gives a simple proof of the fact that $D(z)=\frac{1}{2}(3-z-\sqrt{1-6z+z^2})$. This formula is given a generating function based proof in [11], another (more combinatorial) proof appears in [17], where it is shown, by exhibiting a bijection between OPD-permutations on $\{1,2,\ldots,n\}$ and the lattice paths between $(0,0)$ and $(n-1,n-1)$ that always stay on or below the diagonal with possible steps being $(0,1)$, $(1,0)$ or $(1,1)$, that $d_n=r_{n-1}$, the $(n-1)$th Schröder number.

2.3.2. Parse tree of a P_4-free permutation

If $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ are graphs with $V_1\cap V_2=\phi$, then their union $G_1\cup G_2$ and their join $G_1\times G_2$ are defined by

$$V(G_1\cup G_2)=V_1\cup V_2,\quad E(G_1\cup G_2)=E_1\cup E_2,$$
$$V(G_1\times G_2)=V_1\cup V_2,\quad E(G_1\times G_2)=E_1\cup E_2\cup\{(u,v)\colon u\in V_1, v\in V_2\}.$$

By Lemma 2.2, it is easy to see that cographs may be recursively defined as

- K_1 is a cograph.
- If G_1, G_2 are cographs, then $G_1\cup G_2$ and $G_1\times G_2$ are cographs.
- No graph is a cograph unless it can be obtained using a finite number of applications of the first two conditions.

This recursive definition has been used to associate a unique parse-tree called a cotree with every cograph G (see [3]).

However, since we are interested in P_4-free permutations, we associate with each P_4-free permutation π of $\{1,2,\ldots,n\}$ a unique rooted ordered tree (see [9] for definitions) $T(\pi)$ such that

(i) Every internal node has at least two children.

(ii) The n leaves of $T(\pi)$ are given labels from $\{1,2,\ldots,n\}$ in a specific way and correspond to the n vertices of $G[\pi]$.

(iii) Internal nodes of $T(\pi)$ are given labels from $\{U,X\}$ and accordingly called U-node or X-node; Children of a U-node (resp. X-node) are not U-nodes (resp. X-nodes).

Let π be a P_4-free permutation, so that $G[\pi]$ is a cograph. If $G[\pi]$ has more than one vertex, then by Lemma 2.3 we may write $\pi=\pi_1\pi_2\ldots\pi_k$ $(k>1)$, where

- If G is connected, k is the number of connected components of G^{c}, and $G=G[\pi_1]\times G[\pi_2]\times\cdots\times G[\pi_k]$, each $G[\pi_i]$ is disconnected.
- If G is disconnected, $G[\pi_i]$ for $1\leqslant i\leqslant k$ are the connected components of G.

With each P_4-free permutation π, we associate an ordered tree $T(\pi)$ as follows:

- If $|\pi|=1$ and $\pi=(m)$, $T(\pi)$ consists of the single leaf labelled m.
- If $G[\pi]$ is disconnected, let $\pi=\pi_1\pi_2\ldots\pi_k$ as above. Then $T(\pi)$ consists of a root labelled U with children root($T(\pi_1)$)$\cdots$ root($T(\pi_k)$) in that order from left to right.
- If $G[\pi]$ is connected, let $\pi=\pi_1\pi_2\ldots\pi_k$ as above. Then $T(\pi)$ consists of a root labelled X with children root($T(\pi_1)$)$\cdots$ root($T(\pi_k)$) in that order from left to right.

Note that i,j are adjacent in $G[\pi]$ iff the lowest common ancestor of the leaves labelled $\pi(i)$ and $\pi(j)$ in $T(\pi)$ is a X-node. Define $T_n=\{T(\pi)\,|\,\pi$ is a P_4-free permutation of $(1,2,\ldots,n)\}$ and $P_n=\{\pi\,|\,\pi$ is a P_4-free permutation of $(1,2,\ldots,n)\}$. Then it is easy to see that the above association $T: P_n\to T_n$ is a bijection.

2.3.3. Correspondence between P_4-free and OPD-permutations

Lemma 2.4 (Knuth [11]). *There is a bijection between the set of OPD-permutations on $(1,2,\ldots,n)$ and the set of admissible sequences of length $2n$ on the symbols s,q,x where an admissible sequence of length $2n$ is characterized by*

(i) *There are n x's and n combined s's and q's.*

(ii) *The number of x's must never exceed the combined number of s's and q's reading from the left.*

(iii) *Whenever the number of x's equals the combined number of s's and q's (reading from the left), the next symbol must be a q.*

(iv) *The two symbols xq must never be adjacent in this order.*

Remark. It might appear that conditions (iii) and (iv) above contradict each other, but in fact they do not. The combined effect of these two is that an admissible string S begins with a q and the number of s's and q's strictly exceed the number of x's in any proper prefix of S. For further details, refer [11, Section 2.2.1, Ex 10] or [17].

Lemma 2.5 (Pratt). *Let $G=(V,T,P,S)$ be a context-free grammar with $V=\{S,B\}$, $T=\{s,q,x\}$ and productions $S\to q^n(Bx)^n$, $B\to sq^n(Bx)^{n+1}B$, for all $n\geqslant 0$ and $B\to\varepsilon$. Then G is unambiguous and $U_n=\{y\,|\,y\in L(G)$ and y has n x's$\}$ is precisely the set of all admissible strings in $\{s,q,x\}^*$ of length $2n$ as defined above in Lemma 2.4.*

Theorem 2.2. $|U_n|=|T_n|$.

Proof. We prove this by establishing a bijection between the two sets.

Since G is unambiguous, for each $y\in U_n$ one can associate a unique parse tree $PT(y)$ (see [8]). On deleting from $PT(y)$ the leaves corresponding to the terminals s,q,x and ε, one can get a unique *parse tree* $F(y)\in Q_n$ where Q_n is the set of rooted ordered trees T satisfying:

- The root of T is labelled S, all other nodes of T are labelled B, and T has n leaves.
- All internal nodes except possibly the root have more than one children.

Conversely, it is easy to see that for any such tree $T\in Q_n$, there corresponds a unique string in U_n (the string will be in U_n because T has n leaves all labelled B and any string $w \in (V \cup T)^*$ derived from S without using any ε-productions will have equal number of B's and x's).

Thus it suffices to exhibit a bijection $g : T_n \to Q_n$. Let $T \in T_n$. Define $g(T)$ as follows:

- If the root of T is a U-node, then obtain T' by changing the labels of all nodes (including leaves) of T to B. Then $g(T)$ will consist of a root labelled S with exactly one child root(T').
- If the root of T is a X-node, then obtain $g(T)$ by changing the labels of all the nodes (including the leaves) of T other than the root of T to B and changing the label of the root of T from X to S.

It is a routine matter to check that g is a bijection from T_n onto Q_n. Thus

$$|U_n| = |Q_n| = |T_n|. \qquad \square$$

Theorem 2.3. *The number of P_4-free permutations on $\{1,2,\ldots,n\}$ equals the number of OPD-permutations on $\{1,2,\ldots,n\}$.*

Proof. By Lemma 2.4, $d_n = |U_n|$ and the result follows from Theorem 2.2 and the fact that $|P_n| = |T_n|$. $\square$

Corollary 2.1. *If d_n stands for the number of permutations of $\{1,2,\ldots,n\}$ obtainable using an output-restricted deque, then the generating function $D(z) = \sum_{n\geqslant 0} d_n z^n$ is given in closed form by*

$$D(z) = \frac{3 - z - \sqrt{1 - 6z + z^2}}{2}.$$

Proof. Follows immediately from Theorems 2.1 and 2.3. $\square$

Remark. We have established an interesting bijection between P_4-free permutations and OPD-permutations by exploiting the notions of parse-tree associated with both cographs and context-free grammars. It would be interesting to obtain a more *direct* bijection. Such a bijection may also throw some light upon the *nature* of permutations that can be obtained through an output-restricted deque.

3. Split graphs

An undirected graph G is a split graph if both G and G^c are chordal. The name arises in view of the following:

Theorem 3.1 (Foldes and Hammer [4]). *For an undirected graph $G = (V, E)$, the following are equivalent*:

(i) *G and G^c are chordal.*

(ii) *G contains no induced subgraph isomorphic to $2K_2$, C_4 or C_5.*

(iii) *There is a partition $V = S \cup K$ of the vertex set of G into an independent set S and a clique K.*

In general the partition $V = S \cup K$ of the vertex set of a split graph will not be unique, so the enumeration for the labelled case is interesting and we deal with it first.

3.1. Labelled split graphs

Since the clique number $\omega(G)$ of a split graph G is very useful in determining its structure, we first evaluate $f(n,k)$, the number of labelled split graphs of order n with clique number k.

Theorem 3.2. *For $1 \leqslant k \leqslant n$,*

$$f(n,k) = \binom{n}{k}(2^k - 1)^{n-k} - \binom{n}{k-1}(2^{k-1} - 1)^{n-k+1} + \binom{n}{k-1}2^{(k-1)(n-k+1)} - k\binom{n}{k}2^{(k-1)(n-k)}.$$

Proof. Assume that the vertex set of G is partitioned as $V = S \cup K$ where K is a *maximum* clique, with $|K| = k$. We first observe that $N = \binom{n}{k}(2^k - 1)^{n-k}$ is the number of ways to choose a set K of k vertices from n (labelled) vertices and designate them as the vertices of a maximum clique and allocate subsets ($\neq K$) of these k vertices to the remaining $n-k$ vertices (which will be in S) as their neighborhoods in G.

Now we have to account for those labelled graphs that are counted more than once in N (owing to the non-uniqueness of the splittings as $V = S \cup K$ with $|K| = \omega(G)$). It is not difficult to see that a graph G with a splitting $V = S \cup K$ with $|K| = \omega(G)$ is counted exactly j (>1) times in N iff there exist vertices $v_1 \in K$ and *exactly* $j-1$ vertices $v_2, \ldots, v_j \in S$ such that $N_G(v_1) = N_G(v_2) = \cdots = N_G(v_j) = K - \{v_1\}$. Also these conditions imply that such a $v_1 \in K$ is *unique*. Hence such graphs G (which are counted $j > 1$ times) are *precisely* those that have a vertex subset T such that $|T| = k-1$, $G[T]$ is a clique and $V - T$ is an independent set and a set $P \subseteq V - T$ such that $|P| = j$ and $N_G(v) = T$ for $v \in V - T$ if and only if $v \in P$. It is therefore easy to see that one must subtract $(j-1)\binom{n}{k-1}\binom{n-k+1}{j}(2^{k-1} - 1)^{n+1-k-j}$ from N so that all such graphs are counted exactly once. Thus we get

$$\begin{aligned} f(n,k) &= N - \sum_{j=1}^{n-k+1} \binom{n}{k-1}(j-1)\binom{n-k+1}{j}(2^{k-1} - 1)^{n+1-k-j} \\ &= \binom{n}{k}(2^k - 1)^{n-k} - k\binom{n}{k}2^{(k-1)(n-k)} \\ &\quad + \binom{n}{k-1}[2^{(k-1)(n-k+1)} - (2^{k-1} - 1)^{n-k+1}] \end{aligned}$$

and the result follows. $\square$

Corollary 3.1. *The number s_n of labelled split graphs on n vertices is given by*

$$s_n = 1 + \sum_{k=1}^{n} \binom{n}{k-1} 2^{(k-1)(n-k+1)} - n \sum_{k=1}^{n} \binom{n-1}{k-1} 2^{(k-1)(n-k)}.$$

Proof. Follows from Theorem 3.2 since $s_n = \sum_{k=1}^{n} f(n,k)$. □

3.2. Non-isomorphic split graphs

The symmetry involved in this case necessitates the use of Polya's enumeration theory (see [7,12]). We review some necessary notation. If G (resp. H) are permutation groups acting on X (resp. Y), then the group $G \times H$ acts in the natural way on $X \times Y$ as $(g,h)(x,y) = (gx, hy)$, for all $x \in X, y \in Y$. $Z(G)$ denotes the cycle index of a permutation group G.

Let $f(x; p, q) = \sum_{m=0}^{pq} b_m x^m$ denote the counting series for bicolored graphs where b_m is the number of non-isomorphic bicolored graphs with m edges and $p+q$ vertices with p of them of one color and q of the other color. In the notation of [7], $f(x; p, q)$ is given as follows in [12]:

$$f(x; p, q) = Z(S_p \times S_q; 1 + x),$$

where S_k is the symmetric group of degree k.

Let $[a, b]$ (resp. (a, b)) denote the lcm (resp. gcd) of a and b. Let $j_k(\alpha)$ denote the number of k-cycles of a permutation α. Then as shown in [7],

$$Z(S_m \times S_n) = \frac{1}{m!n!} \sum_{\alpha \in S_m, \beta \in S_n} \prod_{r,t=1}^{m,n} s_{[r,t]}^{(r,t) j_r(\alpha) j_t(\beta)}.$$

Now a split graph $G = (V, E)$ with splitting $V = S \cup K$ with $|S| = p$ and $|K| = q$ can be viewed as a bicolored graph in which the vertices in the clique K get one color and the ones in S get the other color.

Now it is easy to see that $B(x; p, q) = x^{\binom{q}{2}} f(x; p, q)$ is the counting series for split graphs that *admit a splitting* $V = S \cup K$ with $|S| = p, |K| = q$. Define $B(x; p, 0) = 1$, $B(x; 0, q) = x^{\binom{q}{2}}$ and $B(x; p, q) = 0$ if $p < 0$ or $q < 0$.

Theorem 3.3. *Let $SP_k(n; x)$ denote the counting series for non-isomorphic split graphs on n vertices with clique number k $(1 \leqslant k \leqslant n)$. Then $SP_k(n; x) = B(x; n-k, k) - x^k B(x; n-k-1, k)$.*

Proof. Let $G = (V, E)$ be a split graph that *admits* a partition $V = S \cup K$, with $|K| = k$. Then $\omega(G) = k$ unless there exists $v \in S$ such that $N_G(v) = K$. But in that case $H = G - v$ is a split graph with k edges less and that *admits* a splitting $V(H) = S' \cup K'$ where $|S'| = n - k - 1$ and $|K'| = k$. The stated result follows since such graphs G (which are counted in $B(x; n-k, k)$ though their clique-number is not k) are in bijective correspondence with graphs H that *admit* a splitting $V(H) = S \cup K$ with $|S| = n - k - 1$ and $|K| = k$. □

Corollary 3.2. *The counting series $s_n(x)$ for split graphs on n vertices is given by $s_n(x)=\sum_{k=1}^{n} SP_k(n;x)$.*

Example. We illustrate the above corollary for $n=5$. We get

$$s_5(x)=\sum_{k=1}^{5}(B(x;5-k,k)-x^kB(x;4-k,k)).$$

Using $Z(S_n;1+x)=1+x+\cdots+x^n$ and $Z(S_3\times S_2)=1+x+3x^2+3x^3+3x^4+x^5+x^6$ we get $s_5(x)=1+x+x^2+3x^3+3x^4+3x^5+3x^6+3x^7+3x^8+x^9+x^{10}$.

4. C_5-split graphs

We now introduce a class of graphs called C_5-split graphs and proceed to characterize and enumerate them. A graph G is called C_5-split if G has no induced C_4 or $2K_2$. Such a G will also be called $C_4,2K_2$-free. C_5-split graphs form a superclass of split graphs and are in general not perfect (e.g. C_5 — the cycle on five vertices is not perfect). Note that G is C_5-split iff G^c is C_5-split. The following theorem characterizes C_5-split graphs.

Theorem 4.1. *Let $G=(V,E)$ be a $C_4,2K_2$-free graph. Then either G is a split graph or there is a partition of the vertex set of G as $V=C\cup K\cup S$, such that C induces a 5-cycle, K is a clique, S is an independent set, $\{(x,y)\,|\,x\in C, y\in K\}\subset E$ and $\{(x,z)\,|\,x\in C,z\in S\}\cap E=\emptyset$.*

Proof. Let $\omega(G)=p$, and let P be a maximum clique (with $|P|=p$) such that $G[V-P]$ has least number of edges, say k of them. If $k=0$ then G is a split graph. So assume $k\geqslant 1$. Since G has no induced $2K_2$, $G[V-P]$ will have exactly one non-trivial component say $H=(V',E')$, $V'\subseteq V-P$. For $v\in V'$ denote by $d^P(v)$ the number of vertices in P adjacent to v, i.e. $d^P(v)=|N_G(v)\cap P|$.

Let $(a,b)\in E'$ be an edge in H. P being a maximum clique, there must exist *distinct* vertices $d,e\in P$ such that $(a,d)\notin E$, $(b,e)\notin E$. Since G is $C_4,2K_2$-free, exactly one of (a,e) and (b,d) belongs to E. Assume for definiteness, $(b,d)\in E$, $(a,e)\notin E$.

For $x\in P-\{d,e\}$, if $(b,x)\notin E$, then G would have a $2K_2$ induced by $\{a,b,x,e\}$ or a C_4 induced by $\{a,b,x,d\}$ depending on whether $(a,x)\notin E$ or $(a,x)\in E$ respectively. Hence $(b,x)\in E$ and we conclude $N_G(b)\cap P=P-\{e\}$ and $d^P(b)=p-1$. Also clearly $d^P(a)\leqslant p-2$ since $(a,e),(a,d)\notin E$ (see Fig. 1).

Let us call $v\in V'$ *good* if $d^P(v)=p-1$ and *bad* if $d^P(v)<p-1$. By maximality of P, $d^P(v)<p$ for every $v\in V'$ and so every $v\in V'$ is either good or bad (but not both). By the preceding argument a is bad and b is good. Since the arguments were for any edge $(a,b)\in E'$, it follows that every edge of H has one end good and the other end bad. This means that any path in H connecting two good or two bad vertices must be of even length. In particular, H can have no odd cycles. H being an induced subgraph

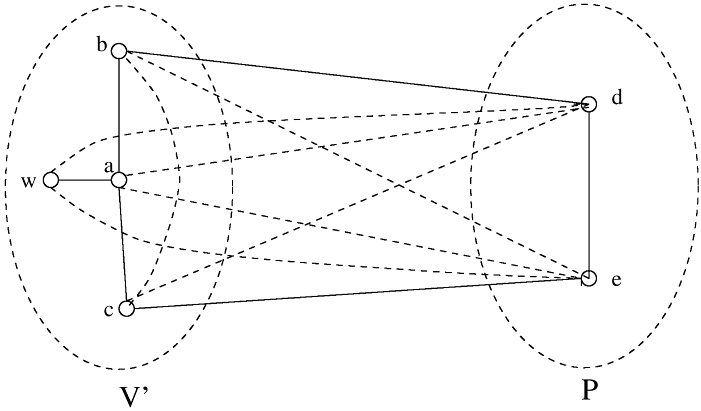

Fig. 1. Structure of a C_5-split graph.

of G, is also $C_4, 2K_2$-free and hence H can have no even cycle as well. Hence H is acyclic and being connected, is a tree. Being $2K_2$-free, H can have no path of length greater than 3, and hence H is a tree with diam$(H) \leqslant 3$.

Consider once again the edge $(a,b) \in E$ with a (resp. b) being a bad (resp. good) vertex. Since $N_G(b) \cap P = P - \{e\}$, $P' = (P \cup \{b\}) - \{e\}$ is also a maximum clique. By our choice of P, $G[V-P']$ can have no fewer edges than $G[V-P]$ which together with the fact that $(a,b) \in E$ but $(a,e) \notin E$ implies that there exists a vertex $c \in V-P$ $(c \neq b)$ such that $(e,c) \in E$ but $(b,c) \notin E$. Now $(a,c) \in E$, for else $\{b,a,c,e\}$ would induce a $2K_2$. Similarly the fact that $\{b,a,c,d\}$ cannot induce a C_4 means that $(c,d) \notin E$ (see Fig. 1, dotted lines indicate absence of an edge).

Thus, for any edge $(a,b) \in E'$ with a being a *bad* vertex, there exists $c \in V'$ $(c \neq b)$ such that $(a,c) \in E'$. Thus, no bad vertex can be a pendant vertex in H. We have seen that diam$(H) \leqslant 3$. Suppose now that diam$(H) = 3$, then there are pendant vertices x, y such that $d_H(x,y) = 3$. But, then x and y would be good vertices connected by a path of odd length, a contradiction. If diam$(H) = 1, H = K_2$ which would imply that H has a pendant vertex that is bad, a contradiction. Hence diam$(H) = 2$ and $H = K_{1,q}$ for some $q \geqslant 2$.

We now prove $q=2$. Indeed if possible let $q > 2$. This means that there exists $w \in V'$ $(w \neq b, c)$ such that $(a,w) \in E'$. Now since G is C_4-free, $(w,d) \notin E$ and $(w,e) \notin E$. So $d^P(w) \leqslant p-2$, i.e w is a bad vertex, a contradiction since $(a,w) \in E'$.

We therefore have $V' = \{a,b,c\}$ and $E' = \{(a,b),(a,c)\}$. Since b,c are good vertices, clearly $N_G(b), N_G(c) \supset P - \{d,e\}$. Also if $z \in P - \{d,e\}$, then $(a,z) \in E$ for otherwise $\{b,a,c,z\}$ would induce a C_4. Hence $N_G(a) \cap P = P - \{d,e\}$. For $y \in V - P - V'$, if $(y,d) \in E$ then $\{a,c,y,d\}$ would induce a $2K_2$, so $(y,d) \notin E$. Similarly $(y,e) \notin E$. Hence $N_G(y) \subseteq P - \{d,e\}$. Now it is easy to see that $C = \{a,b,c,d,e\}$, $K = P - \{d,e\}$ and $S = V - (P \cup \{a,b,c\})$ form the desired partition of V. $\square$

The following result follows at once from Theorem 4.1,

Theorem 4.2. *Let G be a C_5-split graph on n vertices. Then G has at most n maximal cliques and n maximal independent sets. Also if G is not a split graph, then G has exactly five maximum cliques and five maximum independent sets.*

We now turn to enumerative aspects. Since we have already enumerated split graphs, let us now determine y_n (resp. z_n), the number of labelled (resp. non-isomorphic) C_5-split graphs on n vertices that are *not* split graphs.

Theorem 4.3. $y_n = 12\binom{n}{5}\sum_{k=0}^{n-5}\binom{n-5}{k}2^{k(n-5-k)}$.

Proof. Following the terminology of Theorem 4.1, the labels for C can be chosen in $\binom{n}{5}$ ways and after being chosen can be assigned to C_5 in 12 ways (since the automorphism group of C_5 is D_5, the dihedral group of order 5). Let $k = |K|$, then $0 \leqslant k \leqslant n-5$ and K can be chosen in $\binom{n-5}{k}$ ways. Each of the remaining $n-5-k$ vertices comprising S can be assigned any of the 2^k subsets of K as their neighborhoods in the graph in $2^{k(n-5-k)}$ ways. The result now easily follows. □

Recall that $f(x; p, q)$ stands for the counting series for non-isomorphic bicolored graphs on $p+q$ vertices with p of them of one color and q of the other color.

Theorem 4.4. *The counting series $h_n(x)$ for the number of non-isomorphic C_5-split graphs on n vertices that are* not *split graphs is given by*

$$h_n(x) = \sum_{q=0}^{n-5} x^{(q^2+9q+10)/2} f(x; n-5-q, q) = x^5 \sum_{q=0}^{n-5} x^{(q(q+9))/2} Z(S_{n-5-q} \times S_q; 1+x).$$

Proof. Following the terminology of Theorem 4.1, it is easy to see that there is a bijection between the set of C_5-split graphs (that are *not* split graphs) with $|K| = q$ and $|S| = n-5-q$ and the set of bicolored graphs on $n-5$ vertices with q of one color (say R) and $n-5-q$ of the other color (say B). Indeed if G is such a C_5-split graph, obtain $f(G)$ by removing from G all vertices in C and all edges in the clique K and coloring the vertices in K with R and those in S with B. Clearly, f is the desired bijection. The result now easily follows since G has $\binom{q}{2} + 5 + 5q$ more edges than $f(G)$. □

5. Trivially perfect graphs

Let $m(G)$ denote the number of maximal cliques in a graph G. A graph G is said to be trivially perfect if for each $S \subseteq V(G)$, $\alpha(G[S]) = m(G[S])$. These graphs were introduced by Golumbic [5] and are clearly perfect.

Lemma 5.1 (Golumbic [5]). *A graph is trivially perfect iff it has no induced P_4 or C_4.*

Lemma 5.2 (Wolk [21]). *If G is connected and has no induced P_4 or C_4, then G has a vertex of degree $|V(G)|-1$.*

5.1. Labelled trivially perfect graphs

Let P denote a graph theoretic property such that a graph G satisfies P iff every connected component of G satisfies P. For such a property P, let $G_P(x)=\sum_{n=0}^{\infty} g_n x^n/n!$ (resp. $C_P(x)=\sum_{n=0}^{\infty} c_n x^n/n!$) denote the exponential generating function for the labelled (resp. labelled connected) graphs satisfying P. Then it is well-known that (see for example [7])

$$G_P(x)=1+\sum_{n=1}^{\infty}\frac{C_P^n(x)}{n!}=\mathrm{e}^{C_P(x)}.$$

Henceforth, let $G(x)$ (resp. $C(x)$) denote the exponential generating functions for the labelled (resp. labelled connected) trivially perfect graphs. We have $g_1=g_0=c_1=1$ and $c_0=0$.

Lemma 5.3. $C(x)=(1-\mathrm{e}^{-x})G(x)$.

Proof. Let $c_{n,k}$ denote the number of connected trivially perfect graphs G on n vertices with **exactly** k vertices of degree $n-1$. By Lemmas 5.1 and 5.2, we need to consider only k such that $1\leqslant k\leqslant n$ and on removing those k vertices from G, G must become disconnected. Since there are $g_{n-k}-c_{n-k}$ labelled disconnected trivially perfect graphs on $n-k$ vertices, it is easy to see that $c_{n,k}=\binom{n}{k}(g_{n-k}-c_{n-k})$. Hence,

$$C(x)=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\binom{n}{k}(g_{n-k}-c_{n-k})\frac{x^n}{n!}$$

or

$$C(x)=\sum_{k=1}^{\infty}\sum_{r=0}^{\infty}\frac{x^k}{k!}\left(\frac{g_r x^r}{r!}-\frac{c_r x^r}{r!}\right)$$

or

$$C(x)=(\mathrm{e}^x-1)(G(x)-C(x)). \qquad \square$$

Theorem 5.1. *For $n\geqslant 1$,*

(i) $c_n=\sum_{k=1}^{n}\binom{n}{k}(-1)^{k-1}g_{n-k}$.

(ii) $g_n=\sum_{k=1}^{n}\binom{n-1}{k-1}c_k g_{n-k}$.

Proof. (i) Follows from Lemma 5.3.

(ii) Follows easily from Lemma 1.1 and the fact that $G(x)=\mathrm{e}^{C(x)}$. $\square$

Using Theorem 5.1 we calculated c_n and then g_n and we have

$$C(x) = x + \frac{x^2}{2!} + \frac{4x^3}{3!} + \frac{23x^4}{4!} + \frac{181x^5}{5!} + \cdots,$$

$$G(x) = 1 + x + \frac{2x^2}{2!} + \frac{8x^3}{3!} + \frac{49x^4}{4!} + \frac{402x^5}{5!} + \cdots.$$

One can also get an explicit formula for c_n in terms of Stirling numbers of the second kind (see [11, Section 1.2.6] for definitions). For this we relate $C(x)$ to the exponential generating function $T(x)$ for labelled rooted trees which is known to be

$$T(x) = \sum_{n=1}^{\infty} \frac{n^{n-1}x^n}{n!}.$$

It is also known that $T(x)$ satisfies $T(x) = x\mathrm{e}^{T(x)}$ which together with $C(x) = (1-\mathrm{e}^{-x})G(x) = (1-\mathrm{e}^{-x})\mathrm{e}^{C(x)}$ implies that

$$\begin{aligned} C(x) &= T(1-\mathrm{e}^{-x}) \\ &= \sum_{k=1}^{\infty} \frac{k^{k-1}(1-\mathrm{e}^{-x})^k}{k!}. \end{aligned}$$

Now, we have the following formula involving Stirling numbers of the second kind (see [11, Section 1.2.9]):

$$(\mathrm{e}^z - 1)^n = z^n + \frac{1}{n+1}\left\{ \begin{matrix} n+1 \\ n \end{matrix} \right\} z^{n+1} + \cdots = n! \sum_k \left\{ \begin{matrix} k \\ n \end{matrix} \right\} \frac{z^k}{k!}.$$

Thus we have

$$\begin{aligned} C(x) &= \sum_{n=1}^{\infty} n^{n-1} \sum_{k\geqslant 0} \left\{ \begin{matrix} n+k \\ n \end{matrix} \right\} \frac{x^{n+k}}{(n+k)!}(-1)^k \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!} \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^k k^{k-1} \end{aligned}$$

implying

$$c_n = (-1)^n \sum_{k=1}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (-1)^k k^{k-1} \quad \text{for } n \geqslant 1.$$

Remark. The relation $C(x) = T(1-\mathrm{e}^{-x})$ can also be derived directly from Lemma 5.4 of Section 5.2. [2]

Using Lemma 5.4 we want to count equivalence classes of rooted trees, where two rooted trees are considered equivalent if they have the same comparability graph. If $v_1, v_2, \ldots, v_m$ are vertices in a rooted tree such that for $1 \leqslant i < m$, v_{i+1} is the only child

[2] We thank an anonymous referee for this observation.

of v_i, then permuting $v_1, v_2, \ldots, v_m$ arbitrarily yields an equivalent tree, and all equivalent trees can be obtained in this way. Thus we may represent the equivalence class by 'contracting' vertices $v_1, v_2, \ldots, v_m$ to a single vertex with m labels. The maximally contracted tree obtained thus will have the property that no vertex has exactly one child.

Let $U(x)$ be the exponential generating function for rooted (labelled) trees in which no vertex has exactly one child. Since $(\mathrm{e}^x - 1)^k/k!$ the exponential generating function for partitions of a labelled set into k parts, it follows from the properties of exponential generating functions that the exponential generating function for equivalence classes, i.e., $C(x)$, is $U(\mathrm{e}^x - 1)$. Similarly, it can be shown that $T(x)$ and $U(x)$ are related by $T(x) = U(x/(1-x))$, so $U(x) = T(x/(1+x))$. Combining these two formulas we get $C(x) = T(1 - \mathrm{e}^{-x})$ as desired.

5.2. Non-isomorphic trivially perfect graphs

Let f_n (resp. h_n) denote the number of non-isomorphic connected (resp. non-isomorphic) trivially perfect graphs on n vertices. Define $h_0 = f_1 = h_1 = 1$.

Lemma 5.4 (Golumbic [5]). *A connected graph is trivially perfect iff it is a comparability graph whose Hasse diagram is a rooted tree.*

Theorem 5.2. *For $n \geqslant 0$, $h_n = T_{n+1}$ where T_{n+1} stands for the number of rooted trees (non-isomorphic) of order $n+1$.*

Proof. By virtue of Lemmas 5.1 and 5.2, we have $f_k = h_{k-1}$ for $k \geqslant 1$. Hence $h_n = f_{n+1}$. By Lemma 5.4, it is easy to see that $f_{n+1} = T_{n+1}$ and so $h_n = f_{n+1} = T_{n+1}$. $\square$

6. Enumerative aspects of threshold graphs

6.1. Preliminaries

Let $G = (V, E)$ be a simple graph and let $V = \{v_1, v_2, \ldots, v_n\}$. Any subset $X \subseteq V$ corresponds naturally to a characteristic vector $x' = (x_1, x_2, \ldots, x_n)$ where $x_i = 0$ or 1 according as $v_i \in X$ or $v_i \notin X$. G is a threshold graph if there is a linear inequality $\sum_{i=1}^n a_i x_i \leqslant t$ such that $X \subseteq V$ is an independent set iff its characteristic vector satisfies the inequality. Threshold graphs were introduced by Chvatal and Hammer [2], and more details may be found in [14].

Let $G = (V, E)$ be a graph. Let $0 < \delta_1 < \cdots < \delta_m < |V|$ be the distinct degrees of the non-isolated vertices in G. Define $\delta_0 = 0$. The degree partition of G is given by $V = D_0 \cup D_1 \cup \cdots \cup D_m$ where D_i is the set of vertices in G of degree δ_i ($D_i \neq \emptyset$ for $i > 0$).

Let us call a connected threshold graph (i.e. one with $D_0 = \emptyset$) an *odd* or *even* threshold graph according as m is odd or even.

Theorem 6.1 (Golumbic [6]). *Let $G=(V,E)$ be a threshold graph with degree partition $V=\bigcup_{i=0}^{m} D_i$. Define $\delta_{m+1}=|V|-1$. Then $\delta_{i+1}=\delta_i+|D_{m-i}|$ for $i\in\{0,1,\dots,m\}-\{\lfloor m/2\rfloor\}$. Also if $x\in D_i$, $y\in D_j$, $(x\neq y)$,* then *$(x,y)\in E$ iff $i+j>m$.*

Theorem 6.2 (Chvatal and Hammer [2]). *A graph is a threshold graph iff it has no induced $2K_2, P_4$* or *C_4.*

Theorem 6.1 implies that the structure of a threshold graph is entirely determined by the sizes of the parts of the degree partition.

Lemma 6.1. *A threshold graph has at most one non-trivial connected component. Also G is threshold iff G^c is threshold.*

Proof. Both the statements follow at once from Theorem 6.2. □

Lemma 6.2 (Golumbic [6]). *The number of mutually non-isomorphic n-vertex threshold graphs is 2^{n-1}.*

6.2. Labelled threshold graphs

For $n\geqslant 2$, let g_n (resp. c_n) denote the number of labelled threshold graphs (resp. labelled connected threshold graphs) on n vertices. Also define $g_0=g_1=c_0=1$ and $c_1=0$. Let $G(x)$ (resp. $C(x)$) be the exponential generating function of $\{g_n\}_{n\geqslant 0}$ (resp. $\{c_n\}_{n\geqslant 0}$). A closed form expression for $G(x)$ has already been obtained in [1], however we derive it here for the sake of completeness. Moreover, the result in [1] derives a much more general generating function and obtains an expression for $G(x)$ as a by-product, we present a much more direct (and simpler) derivation.

By Theorem 6.1, if G is a connected threshold graph, then G has a vertex of degree $|V|-1$ and hence G^c is disconnected. Hence by Lemma 6.1, $g_n=2c_n$ for $n\geqslant 2$.

Theorem 6.3. $C(x)=(1-x)/(2-e^x)$.

Proof. For $n\geqslant 2$, the number of labelled connected threshold graphs with exactly k vertices of degree $|V|-1$ $(1\leqslant k\leqslant n)$, is clearly given by $\binom{n}{k}\times$ (number of disconnected labelled threshold graphs of order $n-k$). For $n\geqslant 2$, we therefore have

$$c_n=\sum_{k=1}^{n-2}\binom{n}{k}(g_{n-k}-c_{n-k})+\binom{n}{n-1}0+\binom{n}{n}=\sum_{k=1}^{n}\binom{n}{k}c_{n-k}$$

or

$$2c_n = \sum_{k=0}^{n} \binom{n}{k} c_{n-k} \quad \text{for } n \geqslant 2. \tag{4}$$

Hence for $n \geqslant 2$, $2c_n/n! = \sum_{k=0}^{n} c_k/k!(n-k)!$ and thus,

$$2C(x) - 2 = \mathrm{e}^x C(x) - x - 1$$

and the result follows. □

Corollary 6.1. $G(x) = \mathrm{e}^x(1-x)/(2-\mathrm{e}^x)$.

Proof. Since $g_n = 2c_n$ for $n \geqslant 2$, we have $G(x) = 2C(x) + x - 1$ and the result now follows using Theorem 6.3. Eq. (4) along with $g_n = 2c_n$ for $n \geqslant 2$ is also a handy recurrence relation. □

The first few terms of $G(x)$ are

$$G(x) = 1 + x + \frac{2x^2}{2!} + \frac{8x^3}{3!} + \frac{46x^4}{4!} + \frac{332x^5}{5!} + \cdots.$$

6.3. Permutation representations of threshold graphs

By Theorem 6.2, all threshold graphs are cographs and hence also permutation graphs. Let, for $n \geqslant 2, f(n)$ (resp. $g(n)$) denote the number of permutations π of $(1,2,\ldots,n)$ such that $G[\pi]$ is a connected (resp. disconnected) threshold graph. Also define $f(1) = 1$ and $g(1) = 0$. Let $T(n)$ denote the number of permutations π of $(1,2,\ldots,n)$ such that $G[\pi]$ is a threshold graph. Clearly for $n \geqslant 1, T(n) = f(n) + g(n)$. Also by Lemma 6.1, for $n \geqslant 2$, $f(n) = g(n)$.

Theorem 6.4. *For* $n \geqslant 2, T(n) = 4T(n-1) - 2T(n-2)$ *with* $T(0) = T(1) = 1$.

Proof. Let $h(n,k)$ be the number of permutations of $(1,2,\ldots,n)$ such that $G[\pi]$ is the union of k isolated vertices with a connected threshold graph on the remaining $n-k$ vertices, $1 \leqslant k \leqslant n-2$ (by Lemma 6.1, $G[\pi]$ has at most one non-trivial component if it is a threshold graph). Also let $h(n,n-1) = 0$ and $h(n,n) = 1$.

For $1 \leqslant k \leqslant n-2$, using Lemma 2.3, if π is counted in $h(n,k)$, then $\pi = (1,2,\ldots,l)\tau(n-k+l+1,\ldots,n)$ where $0 \leqslant l \leqslant k$ and τ is a permutation of $(l+1,\ldots,n-k+l)$ such that $G[\tau]$ is a connected threshold graph on $n-k$ vertices. Since there are $k+1$ choices for l and $f(n-k)$ choices for τ, we get

$$g(n) = \sum_{k=1}^{n} h(n,k) = 1 + \sum_{k=1}^{n-2} (k+1) f(n-k).$$

For $n \geqslant 2$, $T(n) = 2g(n) = 2f(n)$ and hence

$$T(n) = 2 + \sum_{k=1}^{n-2} (k+1)T(n-k) = 2 + \sum_{k=2}^{n-1} (n-k+1)T(k).$$

We therefore have, for $n \geqslant 4$,

$$T(n) - T(n-1) = \sum_{k=2}^{n-2} T(k) + 2T(n-1),$$

$$T(n-1) - T(n-2) = \sum_{k=2}^{n-3} T(k) + 2T(n-2).$$

From the above two equations

$$T(n) = 4T(n-1) - 2T(n-2) \quad \text{for } n \geqslant 4$$

and it is easy to see that the above is valid for $n = 2, 3$ as well. □

Theorem 6.5. *For* $n \geqslant 0, T(n) = (\sqrt{2})^{n-3}\{(\sqrt{2}+1)^{n-1} + (\sqrt{2}-1)^{n-1}\}$.

Proof. Follows by solving the recurrence relation of Theorem 6.4. □

6.4. Permutation representation of split graphs

In the previous section, we determined $T(n)$, the number of permutations π on $\{1, 2, \ldots, n\}$ such that $G[\pi]$ is a threshold graph. It is well-known that threshold graphs form a proper subclass of split permutation graphs (see, for instance [6]) and we now proceed to determine the asymptotic value of the numbers sp_n, where sp_n stands for the number of permutations π of $\{1, 2, \ldots, n\}$ such that $G[\pi]$ is a split graph.

For a permutation π, denote by $\mathrm{li}(\pi)$ and $\mathrm{ld}(\pi)$ the lengths of the longest increasing and longest decreasing subsequences in π. Clearly $\mathrm{li}(\pi) = \alpha(G[\pi])$ and $\mathrm{ld}(\pi) = \omega(G[\pi])$.

Lemma 6.3. For $n \geqslant 1$, $\mathrm{sp}_n \leqslant \binom{2n}{n}$.

Proof. Let π be a permutation counted in sp_n. For all permutations π, $\mathrm{li}(\pi) + \mathrm{ld}(\pi) \leqslant n+1$ since the longest increasing and decreasing subsequences can have at most one common element. Moreover, since $G[\pi]$ is a split graph, clearly $n \leqslant \mathrm{li}(\pi) + \mathrm{ld}(\pi)$ and π must have an increasing subsequence σ_k of length k and a decreasing subsequence γ_{n-k} (that is disjoint with σ_k) of length $n-k$ for some k, $0 \leqslant k \leqslant n$. The number of permutations π of $\{1, 2, \ldots, n\}$ which have an increasing subsequence σ_k of length k and a disjoint decreasing subsequence of length $n-k$ is clearly bounded above by $\binom{n}{k}^2$ since the elements of σ_k can be chosen in $\binom{n}{k}$ ways and positioned in a further $\binom{n}{k}$ ways. Hence, we have

$$\mathrm{sp}_n \leqslant \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \qquad \square$$

Lemma 6.4. *For* $n \geqslant 1$, $sp_n \geqslant \binom{2n-2}{n-1}$.

Proof. Clearly, if π is such that $\mathrm{li}(\pi) + \mathrm{ld}(\pi) = n+1$, then $G[\pi]$ is a split graph. Let $Y_n = \{\pi : \mathrm{li}(\pi) + \mathrm{ld}(\pi) = n+1\}$. Then $\mathrm{sp}_n \geqslant |Y_n|$.

For $1 \leqslant k \leqslant n$, let $X_{n,k} = \{\pi \in Y_n : \mathrm{l}i(\pi) = k$ and $\mathrm{ld}(\pi) = n-k+1\}$. Since $\mathrm{li}(\pi)$ and $\mathrm{l}d(\pi)$ are respectively the length of the first row and first column of the pair of Young tableaux corresponding to π [18], it follows that for $\pi \in X_{n,k}$, the pair of Young tableaux associated with π (see [10,9,18]) will each have shape $(x_1, x_2, \ldots, x_{n-k+1})$ where $x_i = 1$ for $i > 1$ and $x_1 = k$. If $f(n_1, n_2, \ldots, n_m)$ denotes the number of tableaux formed from $\{1, 2, \ldots, n\}$ that have the given shape $(n_1, n_2, \ldots, n_m)$ where $n_1 + n_2 + \cdots + n_m = n$ and $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_m$, then as shown in [10],

$$f(n_1, n_2, \ldots, n_m) = \frac{\Delta(n_1 + m - 1, n_2 + m - 2, \ldots, n_m) n!}{(n_1 + m - 1)!(n_2 + m - 2)! \cdots n_m!},$$

where $\Delta(x_1, x_2, \ldots, x_m) = \prod_{1 \leqslant i < j \leqslant m} (x_i - x_j)$.

Using the results of [18], we have

$$\begin{aligned} |X_{n,k}| &= (f(k, \underbrace{1, 1, \ldots, 1}_{(n-k)\ 1}))^2 \\ &= \left(\frac{\Delta(n, n-k, n-k-1, \ldots, 1) n!}{n!(n-k)!(n-k-1)! \ldots 1!} \right)^2 \\ &= \binom{n-1}{k-1}^2 \end{aligned}$$

Thus, we get

$$\mathrm{sp}_n \geqslant |Y_n| = \sum_{k=1}^{n} |X_{n,k}| = \sum_{k=1}^{n} \binom{n-1}{k-1}^2 = \binom{2n-2}{n-1}. \qquad \square$$

Theorem 6.6. *The asymptotic value of the number of permutations* π *of* $\{1, 2, \ldots, n\}$ *such that* $G[\pi]$ *is a split graph is* $\Theta(4^n/\sqrt{n})$.

Proof. By Lemmas 6.3 and 6.4,

$$\frac{1}{4}\binom{2n}{n} \leqslant \binom{2n-2}{n-1} \leqslant \mathrm{sp}_n \leqslant \binom{2n}{n}$$

and hence $\mathrm{sp}_n \in \Theta\left(\binom{2n}{n}\right) = \Theta(4^n/\sqrt{n})$. $\square$

Remark. Threshold graphs form a subclass of split permutation graphs (see [6]) and we obtained a closed form expression for the permutation representations of threshold graphs. Similarly, it would be interesting to determine the exact number of permutations π of $\{1, 2, \ldots, n\}$ such that $G[\pi]$ is a split graph.

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